Eur. Phys. J. B **30**, 543–549 (2002)
DOI: 10.1140/epjb/e2002-00412-6

THE EUROPEAN PHYSICAL JOURNAL B

Generalized thermostatistics based on the Sharma-Mittal entropy and escort mean values

T.D. Frank[1,a] and A.R. Plastino[2]

[1] Institute for Theoretical Physics, University of Münster Wilhelm-Klemm-Strasse 9, 48149 Münster, Germany
[2] Faculty of Astronomy and Geophysics, National University La Plata, C.C. 727, 1900 La Plata, Argentina

Received 27 June 2002
Published online 31 December 2002 –

Abstract. A generalized thermostatistics is developed for an entropy measure introduced by Sharma and Mittal. A maximum-entropy scheme involving the maximization of the Sharma and Mittal entropy under appropriate constraints expressed as escort mean values is advanced. Maximum-entropy distributions exhibiting a power law behavior in the asymptotic limit are obtained. Thus, results previously derived for the Renyi entropy and the Tsallis entropy are generalized. In addition, it is shown that for almost deterministic systems among all possible composable entropies with kernels that are described by power laws the Sharma-Mittal entropy is the only entropy measure that gives rise to a thermostatistics based on escort mean values and admitting of a partition function.

PACS. 05.20.-y Classical statistical mechanics – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion

1 Introduction

Power law distributions are ubiquitous in Nature. They can be observed in a variety of disciplines ranging from the hydrodynamics of fully developed turbulence [1–3] to urban development [4] and from biophysics [5,6] to economics [7,8] to neurophysics [9]. They often occur in the context of Lévy flights [10–14] and are characteristic for systems exhibiting self-organized criticality [15]. Power law distributions have recently been related to generalized thermostatistics [3,7,16–20], namely, the thermostatistics based on the Renyi entropy [21,22] and the Tsallis entropy [23].

Power law distributions have also been discussed in connection with the Sharma and Mittal entropy, which unifies and generalizes the Renyi entropy and the Tsallis entropy [24]. Moreover, in the context of the specific heat of nonextensive systems, a two-parametric entropy measure similar to the Sharma-Mittal entropy has recently been studied [25]. As we are going to show in Section 2, the Sharma and Mittal measure also comprises as particular instances other nonstandard entropic measures that have been applied to physical problems, such as the nonadditive measure recently advanced by Landsberg and Vedral in connection with generalized channel capacities [26]. A thermostatistical formalism for the Sharma and Mittal entropy, using standard linear constraints, was considered in [24]. However, recent developments in the field of nonextensive thermostatistics [27] suggest that, within the context of systems described by either the Renyi entropy [18,19] or the Tsallis entropy [28], escort mean values provide the appropriate type of constraint to be used. For historical reasons, generalized thermostatics based upon ordinary linear constraints are referred to as "first choice thermostatistics". In contrast, thermostatistical descriptions involving constraints of the form $U = \sum_{i=1}^{N} \epsilon_i p_i^q$ (where the p_is describe a complete set of probabilities) are referred to as "second choice thermostatistics". Finally, when escort mean values are used, generalized thermostatistics are called "third choice thermostatistics" [28]. Among other desirable properties, thermostatistics based on escort mean values yield distributions that are independent of the origins of energy scales [29]. Therefore, at issue is to derive a third choice thermostatistics for the Sharma-Mittal entropy and thus both unify and generalize the earlier works on the Renyi entropy and the Tsallis entropy.

In the present article, we will present such a unifying thermostatistics. Subsequently, we will establish a general relationship between composable entropies with entropy kernels given by power laws and partition functions. Finally, we will show that for the simplest composable entropy (or for almost deterministic systems) a thermostatistics based on a partition function can only be obtained in the case of the Sharma-Mittal entropy.

[a] e-mail: tdfrank@uni-muenster.de

2 Sharma-Mittal entropy and escort mean values

2.1 Sharma-Mittal entropy

In line with a study by Sharma and Mittal [30], we consider the two-parametric entropy

$$S_s(q)(\{p_i\}_{i=1,\dots,n}) := \frac{1-\left(\sum_{i=1}^N p_i^s\right)^{(q-1)/(s-1)}}{q-1}, \quad q,s>0\ ,\ q,s\neq 1 \quad (1)$$

$$S_s(q=1)((\{p_i\}_{i=1,\dots,n}) :=^R S_s = \frac{1}{1-s}\ln\sum_{i=1}^N p_i^s, \quad s>0\ ,\ s\neq 1 \quad (2)$$

$$S_{s=1}(q)((\{p_i\}_{i=1,\dots,n}) :=^G S_s = \frac{1-\exp\left\{(q-1)\sum_{i=1}^N p_i\ln p_i\right\}}{q-1}, \quad q>0\ ,\ q\neq 1\ , \quad (3)$$

$$S_{s=1}(q=1)((\{p_i\}_{i=1,\dots,n}) :=^{BGS} S = -\sum_{i=1}^N p_i\ln p_i \quad (4)$$

for a complete set of probabilities p_i (*i.e.*, $\sum_{i=1}^N p_i = 1$). Here, ${}^RS_s$ denotes the Renyi entropy [21,22], ${}^{BGS}S$ stands for the Boltzmann-Gibbs-Shannon entropy [31,32], and it has been suggested to call ${}^GS_s$ the Gaussian entropy [24]. It can be shown [30] that the following limiting cases hold:

$$\lim_{q\to 1} S_s(q) = {}^RS_s\ ,\quad \lim_{s\to 1} S_s(q) = {}^GS(q)\ ,$$
$$\lim_{q\to 1}\lim_{s\to 1} S_s(q) = \lim_{s\to 1}\lim_{q\to 1} S_s(q) = {}^{BGS}S\ . \quad (5)$$

Furthermore, for two statistically independent systems given by the probability distributions $\{p_i\}_{i=1,\dots,N}$ and $\{p'_i\}_{i=1,\dots,N}$, the entropies $S_s(q)$ and ${}^GS(q)$ satisfy

$$S_s(q)(\{p_ip'_k\}) = S_s(q)(\{p_i\}) + S_s(q)(\{p'_k\}) + (1-q)\ S_s(q)(\{p_i\})\,S_s(q)(\{p'_k\})\ ,$$
$${}^GS(q)(\{p_ip'_k\}) =^G S(q)(\{p_i\}) + {}^GS(q)(\{p'_k\}) + (1-q)\,{}^GS(q)(\{p_i\})\,{}^GS(q)(\{p'_k\})\cdot \quad (6)$$

Consequently, for $q\neq 1$ the entropy $S_s(q)$ is nonextensive and q can be regarded as a measure of the degree of nonextensivity [1]. For $q=s$ the entropy $S_s(q)$ recovers the Tsallis entropy ${}^TS(q)$ [23] which reads

$$S_q(q)((\{p_i\}_{i=1,\dots,n}) = {}^TS(q) = \frac{1-\left(\sum_{i=1}^N p_i^q\right)}{q-1} \quad (7)$$

and is central to the development of a nonextensive generalized thermodynamics [7,27,33,34]. When $s+q=2$ the Sharma-Mittal entropy $S_s(q)$ reduces (up to a multiplicative constant) to the information measure $S_s^{(u)} = \frac{k}{1-s}(1-[\sum p_i^s]^{-1})$, recently introduced by Landsberg and Vedral [26] in connection with generalized channel capacities.

[1] One caveat. In an earlier study [24] the parameters s and q were interchanged.

For both the Renyi entropy and the Tsallis entropy a thermostatistical formalism has been developed [18,19,28] using the energy constraint

$$U = \frac{\sum_{i=1}^N p_i^\nu\,\epsilon_i}{\sum_{i=1}^N p_i^\nu}\ , \quad (8)$$

where ϵ_i describe the energy levels of the system under consideration with $\nu=s$ for ${}^RS(s)$ and $\nu=q$ for ${}^TS(q)$. Mean values of the form (8) are known as *escort mean values* [27]. Before being applied to the field of nonextensive thermostatistics, escort mean values had found important applications in connection with the thermostatistics of multifractals [22]. The problem we are going to tackle here is how to derive a thermostatistics based on escort constraints (that is, "a third choice thermostatistics") for the two-parametric entropy $S_s(q)$. In [24] it has been suggested to related the energy constraint (8) to the parameter s that describes the deviations of maximum-entropy distributions of $S_s(q)$ from Boltzmann distributions. This suggestion is in line with the observation that the Renyi entropy ${}^RS_s = S_s(q)$ and the Tsallis entropy ${}^TS(q) = S_{s=q}(q)$ have the parameter s in common when interpreting them as special cases of the Sharma-Mittal entropy $S_s(q)$. Consequently, we consider now the generalized canonical ensemble described by the entropy $S_s(q)$, the energy constraint (8) with $\nu=s$ and the normalization condition $\sum_{i=1}^N p_i = 1$. To derive the maximum-entropy distribution of this ensemble, we introduce the Lagrange parameters α and β and the function

$$I(\{p_i\}) := S_s(q)(\{p_i\}) + \alpha\left(1-\sum_{i=1}^N p_i\right) + \beta\left(U - \frac{\sum_{i=1}^N p_i^s\,\epsilon_i}{\sum_{i=1}^N p_i^s}\right) \quad (9)$$

and require that the variation δI vanishes for all perturbations δp_i of the maximum-entropy distribution. Thus, the maximum-entropy distribution is found as

$$p_i = \frac{1}{Z}\left[1-\frac{(1-s)\beta\,(\epsilon_i-U)}{\left[\sum_i p_i^s\right]^{(q-1)/(s-1)}}\right]_+^{1/(1-s)} \quad (10)$$

with $[X]_+ := \max\{X,0\}$ defining a cut-off condition [23,24] and Z given by

$$Z := \left[\frac{\alpha(1-s)}{s}\left[\sum_i p_i^s\right]^{(s-q)/(s-1)}\right]^{1/(1-s)}\ . \quad (11)$$

Equation (10) can equivalently be expressed as

$$p_i = \frac{1}{Z}\left[1 - \frac{(1-s)\beta\,(\epsilon_i - U)}{Z^{1-q}}\right]_+^{1/(1-s)} . \qquad (12)$$

To derive equation (12), we consider the product $p_i^{1-s} Z^{1-s}$. Then, by virtue of equation (10) we obtain

$$p_i^{1-s} Z^{1-s} = \left[1 - \frac{(1-s)\beta\,(\epsilon_i - U)}{\left[\sum_i p_i^s\right]^{(q-1)/(s-1)}}\right]_+ . \qquad (13)$$

Multiplying equation (13) with p_i^s and summing up all terms $p_i Z^{1-s}$ we obtain

$$Z^{1-s} = \sum_{i=1}^{N} p_i^s \qquad (14)$$

because of the constraint $\sum_{i=1}^N p_i^s\,\epsilon_i = U \sum_{i=1}^N p_i^s$. Equation (14) was previously found for the Renyi entropy [19] and for the Tsallis entropy ${}^T S(q) = S_{s=q}(q)$ [28]. Substituting equation (14) into equation (10) gives us the result (12). We can read off from equation (12) that in the asymptotic limit, that is, for $\epsilon_i - U \gg Z^{1-q}/(1-s)\beta$, the maximum-entropy distribution p_i is described by a power-law distribution: $p_i \propto (\epsilon_i - U)^\mu$ with $\mu = 1/(1-s)$. Moreover, the maximum-entropy distribution of $S_s(q)$ includes as special cases the maximum-entropy distributions that have been derived in earlier studies for the Renyi entropy [18,19] and the Tsallis entropy [28].

Our next objective is to related the maximum-entropy distribution p_i to a partition function $\tilde{Z}$. To this end, we first introduce for $x > 0$ the generalized logarithmic function [28]

$$\ln_q(x) := \frac{1 - x^{1-q}}{q-1}, \quad q>0\ , \quad q \neq 1;$$
$$\ln_{q=1}(x) := \ln(x). \qquad (15)$$

Equation (1) can then be written as

$$S_s(q) = \ln_q\left(\left[\sum_i p_i^s\right]^{\frac{1}{1-s}}\right) . \qquad (16)$$

Consequently, by means of equation (14) we can express the entropy $S_s(q)$ of the maximum-entropy distribution as

$$S_s(q)(\max) = \ln_q Z\ , \qquad (17)$$

which yields for the Tsallis entropy $S_q(q) = {}^T S(q) = \ln_q Z$ [28] and for the Renyi entropy $S_s(q=1) = {}^R S_s = \ln Z$ [19]. In line with earlier work [19,28], we define now implicitly the partition function $\tilde{Z}$ by

$$\ln_q \tilde{Z} := \ln_q Z - \beta U\ . \qquad (18)$$

Then the free energy $F_s(q)(\{p_i\}_{i=1,\ldots,N})$ defined by

$$F_s(q)(\{p_i\}) := U - T S_s(q)(\{p_i\}) = U - \frac{1}{\beta} S_s(q)(\{p_i\})\ , \qquad (19)$$

where $\beta = \partial S/\partial U = 1/T$ is regarded as an inverse temperature measure [35–38], can be written as

$$F_s(q)(\max) = -\frac{1}{\beta}\ln_q \tilde{Z} \qquad (20)$$

for the maximum-entropy distribution (12). Finally, it can be shown (see Appendix A) that the mean energy U can be computed from $\tilde{Z}$ by

$$U = -\frac{\partial}{\partial\beta}\ln_q \tilde{Z}\ . \qquad (21)$$

On account of the relations (20) and (21), the function $\tilde{Z}$ can indeed be considered as the generalized partition function of the third choice thermostatistics based on the Sharma-Mittal entropy $S_s(q)$. In addition, equations (20, 21) recover previously derived results for the entropies ${}^R S_s$ [19] and ${}^T S(q)$ [28]. Finally, from equations (19, ... , 21) it follows that $\partial S_s(q)/\partial\beta = \beta\,\partial U/\partial\beta$ and, consequently, we can compute the specific heat capacity $C_s(q) := T\,\partial S_s(q)/\partial T$ for the maximum-entropy distribution (12) according to

$$C_s(q) = T\,\frac{\partial S_s(q)}{\partial T} = \frac{\partial U}{\partial T} = -T\,\frac{\partial^2 F_s(q)}{\partial T^2}\ , \qquad (22)$$

cf. also [19,28].

2.2 Composable entropies with kernels described by power laws

The relationship (17) between the entropy of a maximum-entropy distribution and the normalization constant of that distribution can be viewed as the pivot element for the development of a thermostatistics based on a partition function (*i.e.*, for deriving Eqs. (20, 21)). In what follows, we show that this relation can be generalized for composable entropies [7] given by

$${}^B S_s := B\left(\sum_{i=1}^N \tilde{S}(p_i)\right) = B\left(\sum_{i=1}^N p_i^s\right)\ , \qquad (23)$$

and

$${}^B S_s(\{p_i p'_k\}) = {}^B S_s(\{p_i\}) + {}^B S_s(\{p'_k\}) + N\left({}^B S_s(\{p_i\}),\, {}^B S_s(\{p'_k\})\right) \qquad (24)$$

where $B(z)$ is a differentiable function describing an overall entropy scale [39], $\tilde{S}(p_i)$ denotes the entropy kernel, and $N(x,y)$ is a differentiable function $N \in C^1(\mathbb{R}\times\mathbb{R})$

with $N(u,v) = N(v,u)$ and $N(u,0) = N(0,v) = 0$. Alternative to the approach in the previous section, the departure point now is the relation

$$\rho(m)\sum_{i=1}^{N}\left[\tilde{S}(p_i) - p_i\frac{\mathrm{d}\tilde{S}}{\mathrm{d}p_i}\right] = K + \frac{\partial}{\partial v}N(S,v)\bigg|_K$$
$$= K + n(S) \tag{25}$$

with

$$\rho(z) := \mathrm{d}B/\,\mathrm{d}z\ ,$$
$$m := \sum_{i=1}^{N}\tilde{S}(p_i)\ ,$$
$$K := \rho(\tilde{S}(1))\left(\tilde{S}(1) - \frac{\mathrm{d}\tilde{S}(p_i)}{\mathrm{d}p_i}\bigg|_{p_i=1}\right)\ ,$$
$$n(z) := \frac{\partial}{\partial v}N(z,v)\bigg|_{v=K}\ . \tag{26}$$

Equation (25) holds for any composable entropy $S = B(\sum_i \tilde{S}(p_i))$ [40]. In particular, for $S = {}^B S_s$ and $\tilde{S}(p_i) = p_i^s$ equation (25) reduces to

$$n({}^B S_s) = (1-s)\left[\rho\left(\sum_i p_i^s\right)\sum_i p_i^s - \rho(1)\right]. \tag{27}$$

If $n(z)$ is invertible and $\sum_i p_i^s$ can be expressed for the maximum-entropy distribution of ${}^B S_s$ in terms of a normalization factor Z, then equation (27) generalizes equation (17). Replacing $S_s(q)$ in equation (9) by ${}^B S_s$, we can compute the maximum-entropy distribution by requiring that the variation of the function I vanishes and obtain

$$p_i = \frac{1}{Z}\left[1 - \frac{\beta\,(\epsilon_i - U)}{\rho\left(\sum_i p_i^s\right)\sum_i p_i^s}\right]_+^{1/(1-s)} \tag{28}$$

with

$$Z := \left[\frac{\alpha}{s\,\rho(\sum_i p_i^s)}\right]^{1/(1-s)}\ . \tag{29}$$

Just as for equation (10), from equation (28) it follows the equivalence (14). Consequently, equations (27, 28) read

$$n({}^B S_s(\max)) = (1-s)\left[Z^{1-s}\rho\big(Z^{1-s}\big) - \rho(1)\right] \tag{30}$$

and

$$p_i = \frac{1}{Z}\left[1 - \frac{\beta\,(\epsilon_i - U)}{Z^{1-s}\rho\big(Z^{1-s}\big)}\right]_+^{1/(1-s)}\ , \tag{31}$$

respectively. Assuming the existence of an inverse $n^{-1}(z)$ of $n(z)$, we obtain the final result

$${}^B S_s(\max) = L(Z),$$
$$L(Z) := n^{-1}\big((1-s)\left[Z^{1-s}\rho\big(Z^{1-s}\big) - \rho(1)\right]\big)\,. \tag{32}$$

Example

To illustrate the power of equations (30, ..., 32), we consider again the Sharma-Mittal entropy (1). If we identify $B(z)$ with $B(z) = [1 - z^{(q-1)/(s-1)}]/\,[q-1]$ which implies that $\rho(z) = z^{(q-s)/(s-1)}/\,[1-s]$, $\rho(1) = 1/\,[1-s]$, and $\rho(Z^{1-s}) = Z^{s-q}/\,[1-s]$, we realize that the maximum-entropy distribution (31) becomes the distribution (12) derived in Section 2.1. Furthermore, the right hand side of equation (30) reads $RHS = Z^{1-q} - 1$. Since $N(u,v) = (1-q)uv \Rightarrow n(z) = (1-q)z$, the left hand side of equation (30) reads $LHS = (1-q)\,{}^B S_s$. Consequently, for ${}^B S_s = S_s(q)$ equation (30) agrees with equation (17).

2.3 Relevance of the Sharma-Mittal entropy

Entropies are usually defined as measures that describe the degree of disorder of a system. Moreover, for purely deterministic states (*i.e.*, for $p_i = \delta_{i,k}$, where $\delta_{i,k}$ denotes the Kronecker symbol) entropy measures vanish and increase continuously when the amount of disorder increases. Therefore, for almost deterministic systems described by composable entropies given by equations (23, 24) we can interpret ${}^B S_s$ as a small parameter and expand $N(u,v)$ into a Taylor series with respect to ${}^B S_s$. Taking the condition $N(u,0) = N(0,v) = 0$ into account, in the lowest order we thus obtain $N(u,v) = N_{1,1}uv$, where $N_{1,1}$ measures the degree of nonextensivity. Consequently, using $N_{1,1} = (1-q)$, composable entropies (23) of almost deterministic systems satisfy in the lowest order approximation

$${}^B S_s(\{p_i p_k'\}) = {}^B S_s(\{p_i\}) + {}^B S_s(\{p_k'\})$$
$$+ (1-q)\,{}^B S_s(\{p_i\})\,{}^B S_s(\{p_k'\}). \tag{33}$$

Alternatively, equation (33) may be regarded as the simplest composable entropy. At issue is now to find all possible entropies that satisfy equations (23) and (33) and facilitate thermodynamic descriptions in terms of partition functions. Put differently, if we require that relations similar to equations (20, 21) hold, to which extent does this constraint restrict the choice of the scaling function $B(z)$?

First of all, for $N(u,v) = (1-q)uv$ equation (30) reads

$${}^B S_s(\max) = L(Z) = \frac{1-s}{1-q}\left[Z^{1-s}\rho\big(Z^{1-s}\big) - \rho(1)\right]\ . \tag{34}$$

Next, we require that the following identities hold: ${}^B F_s(\max) := U - \beta^{-1}\,{}^B S_s(\max) = -\beta^{-1}L(\tilde{Z})$ and $U = -\,\partial L(\tilde{Z})\big/\,\partial\beta$ with $L(\tilde{Z}) := L(Z) - \beta U$, *cf.* equations (18, ..., 21). This requirement leads to

$$\beta\frac{\partial U}{\partial\beta} = \frac{\partial\,{}^B S_s(\max)}{\partial\beta}\ , \tag{35}$$

which can be checked by differentiating ${}^B F_s$ with respect to β. Conversely, if equation (35) holds we can indeed derive the above relations between ${}^B F_s(\max)$, U, and $L(\tilde{Z})$.

Then, it is clear from equations (34, 35), that the function $\rho(z)$ must satisfy the differential equation

$$\beta\frac{\partial U}{\partial\beta} = \frac{\partial L(Z)}{\partial\beta} = \frac{1-s}{1-q}\,\frac{\partial}{\partial\beta}\left[Z^{1-s}\rho\big(Z^{1-s}\big) - \rho(1)\right] . \quad (36)$$

Using equation (31), we can express Z as $Z = \sum_i[\ldots]_+^{1/(1-s)}$. Then, we differentiate Z with respect to β. In doing so, we need to carry out a computation similar to that presented in the Appendix and obtain

$$\frac{\partial Z}{\partial\beta} = \frac{\beta\, Z^s}{(1-s)\rho(Z^{1-s})}\,\frac{\partial U}{\partial\beta} \cdot \quad (37)$$

From equation (34) it follows that

$$\frac{\partial L(Z)}{\partial\beta} = \frac{1-s}{1-q}\,\frac{1}{\rho(z)}\,\frac{\partial z\rho(z)}{\partial z}\bigg|_{z=Z^{1-s}}\,\beta\frac{\partial U}{\partial\beta} \cdot \quad (38)$$

Comparing equations (36) and (38), we need to solve now the differential equation

$$1 = \frac{1-s}{1-q}\,\frac{1}{\rho(z)}\,\frac{\partial z\rho(z)}{\partial z}\bigg|_{z=Z^{1-s}} , \quad (39)$$

which gives us

$$\rho(z) = \frac{\mathrm{d}B}{\mathrm{d}z} = \rho_0(s,q)\, z^{(s-q)/(1-q)}\Big|_{z=Z^{1-s}} , \quad (40)$$

where $\rho_0(s,q)$ denotes an integration constant that may depend on the parameters s and q. Integrating equation (40), we find

$$\begin{aligned}
{}^{B}S_s(\{p_i\}) &= B\left(\sum_i p_i^s\right)\\
&= B_0(s,q)+\rho_0(s,q)\frac{1-s}{1-q}\,z^{(1-q)/(1-s)}\Big|_{z=Z^{1-s}=\sum_i p_i^s}\\
&= B_0(s,q) + \rho_0(s,q)\frac{1-s}{1-q}\left[\sum_i p_i^s\right]^{(1-q)/(1-s)}\\
&= (1-s)\rho_0(s,q)\ln_q\left\{\left[\sum_i p_i^s\right]^{1/(1-s)}\right\}\\
&\quad + B_0(s,q) + \rho_0(s,q)\frac{1-s}{1-q} \qquad (41)\\
&= (1-s)\rho_0(s,q)S_s(q) + B_0(s,q) + \rho_0(s,q)\frac{1-s}{1-q} \cdot \qquad (42)
\end{aligned}$$

Note that in order to obtain equation (42) from equation (41) we used equation (16). Obviously, the entropy ${}^{B}S_s$ agrees with the Sharma-Mittal entropy apart from some factors depending solely on the parameters s and q. We may determine the integration constants $B_0(s,q)$ and $\rho_0(s,q)$ by requiring that in the limits $\lim_{s\to1}\lim_{q\to1}$ and $\lim_{q\to1}\lim_{s\to1}$ the entropy ${}^{B}S_s$ converges to the Boltzmann-Gibbs-Shannon entropy ${}^{BGS}S$. These constraints are satisfied for $\lim_{s\to1}(1-s)\rho_0(s,q=1) = 1$ and $\lim_{q\to1}[B_0(s=1,q)+(1-q)^{-1}] = 0$. The simplest choice of ρ_0 and B_0 which is in agreement with these limiting cases, namely, $\rho_0(s,q) := (1-s)^{-1}$ and $B_0(s,q) := (q-1)^{-1}$, leads to the Sharma-Mittal entropy ${}^{B}S_s = S_s(q)$.

2.4 The Sharma-Mittal thermostatistics and the "P-picture"

A maximum entropy principle based upon escort constraints can be reformulated in terms of standard linear constraints [27,29]. Let us consider an entropy functional $S[\{p_i\}]$, and the maximum entropy variational problem

$$\delta\left\{S[\{p_i\}]+\alpha\left(1-\sum_{i=1}^{N}p_i\right)+\beta\left(U-\frac{\sum_{i=1}^{N}p_i^s\epsilon_i}{\sum_{i=1}^{N}p_i^s}\right)\right\} = 0. \quad (43)$$

We introduce now the new set of probabilities

$$P_i \;=\; \frac{p_i^s}{\left(\sum_{i=1}^{N}p_i^s\right)} \cdot \quad (44)$$

It is clear that the probabilities P_i are appropriately normalized. On the other hand, assuming that the original p_i's were normalized as well, it is easy to verify that

$$\sum_{i=1}^{N}p_i^s \;=\; \left(\sum_{i=1}^{N}P_i^{1/s}\right)^{-s} . \quad (45)$$

Using this last equation, it is possible to invert the transformation (44), and express the set of probabilities $\{p_i\}$ in terms of the set $\{P_i\}$,

$$p_i \;=\; \frac{P_i^{1/s}}{\left(\sum_{i=1}^{N}P_i^{1/s}\right)} \cdot \quad (46)$$

From equations (44) and (46) we can conclude that the transformation (44) constitutes a *one to one* map of the N-simplex on itself.

Now, we can re-express the entropy functional $S[p]$ in terms of the new probabilities P_i. That is, we can introduce a new functional S^* such that

$$S[\{p_i\}] \;=\; S^*[\{P_i\}] \,. \quad (47)$$

The maximum entropy variational problem (43) can be recast in terms of the set of probabilities $\{P_i\}$ as

$$\delta\left\{S^*[\{P_i\}]+\alpha\left(1-\sum_{i=1}^{N}P_i\right)+\beta\left(U-\sum_{i=1}^{N}P_i\epsilon_i\right)\right\} = 0. \quad (48)$$

It is important to realize that the two variational problems (43) and (48) are completely equivalent. Each one is just a re-formulation of the other one, obtained by recourse to the (invertible) change of variables defined by equations (44) and (46). The two formulations (43) and (48) differ, however, in an important point: the first one is expressed in terms of escort mean values, while the second one involves only standard, linear mean values (also, as noted in [41], the concavity properties of the entropy in the P-picture may be different from the corresponding properties in the p-picture). The re-formulation (48) was first considered in connection with the third choice thermostatistics associated with the Tsallis entropy. Within the context of nonextensive thermostatistics, the formulation (48) is referred to as the "P-picture" [27].

If Tsallis functional S_q is replaced in the left hand side of equation (47), then the right hand side of that equations yields the new measure (putting $s = q$ in (44) and (46))

$$S_q^* = \frac{1}{q-1}\left[1 - \left(\sum_{i=1}^{N} P_i^{1/q}\right)^{-q}\right] . \qquad (49)$$

Notice that the measure $S_q^*[\{P_i\}]$ belongs to the Sharma-Mittal family of entropies (1). Indeed, the functional $S_q^*[\{P_i\}]$ is recovered if we put $s = 1/q$ in (1). The entropy (49), which we have shown here to be a particular instance of the Sharma Mittal entropy, has recently been applied to the analysis of complex signals.

It is a natural question to ask now what happens if we apply the transformation defined by (44) and (46) to the third choice thermostatistics based upon a general member of the Sharma and Mittal family of measures (1). If we replace the (general) Sharma Mittal entropy in the left hand side of equation (47), we have,

$$\begin{aligned} S_s(q)[\{p_i\}] &= \frac{1 - \left(\sum_{i=1}^{N} p_i^s\right)^{(q-1)/(s-1)}}{q-1} \\ &= \frac{1 - \left(\sum_{i=1}^{N} P_i^{1/s}\right)^{-s\left(\frac{q-1}{s-1}\right)}}{q-1} \\ &= \frac{1 - \left(\sum_{i=1}^{N} P_i^{1/s}\right)^{\left(\frac{q-1}{(1/s)-1}\right)}}{q-1} \cdot \end{aligned} \qquad (50)$$

Consequently, the right hand side of equation (47) adopts the form

$$S^*[\{P_i\}] = S_{s^*}(q)[\{P_i\}] , \qquad (51)$$

where

$$s^* = \frac{1}{s} \cdot \qquad (52)$$

This means that the "P-picture" representation of the third choice thermostatistics derived from a member of the Sharma-Mittal entropic family is described, after an appropriate redefinition of the s-parameter given by (52), by another member of the Sharma-Mittal family. In other words, *a maximum entropy principle based on the entropy $S_s(q)[\{p_i\}]$ and formulated in terms of escort constraints, is equivalent to a maximum entropy principle based on the measure $S_{s^*}(q)[\{P_i\}]$, with s^* given by (52), and involving only ordinary linear constraints.*

On the light of our previous discussion, we can say that the Sharma-Mittal family of entropies is "closed" under the action of the transformations (44) and (46). This remarkable fact constitutes further evidence suggesting that the generalized thermostatistical formalism derived here from the Sharma-Mittal bi-parametric measure may be useful for obtaining new insights on various issues raised by the Renyi and Tsallis thermostatistics.

3 Conclusions

We have developed a third choice thermostatistics for nonextensive systems with equilibrium power-law distributions based on an entropy proposed by Sharma and Mittal that depends on two parameters. One of these parameters is related to the exponent that characterizes the maximum-entropy power-law distributions of the Sharma-Mittal entropy. In this sense, it describes the deviation of the concomitant equilibrium distributions from Boltzmann distributions. The other parameter can be regarded as a measure of the degree of nonextensivity exhibited by the physical system under study. Consequently, in the context of the Sharma-Mittal entropy, effects due to shape distortion and nonextensivity can be studied separately. The Tsallis entropy "lives" on the diagonal of the two-dimensional parameter space associated with the Sharma-Mittal entropy. As a consequence, the Sharma-Mittal entropy can act as a magnifying glass for the Tsallis entropy and may reveal to which extent phenomena observed for this entropy measure are related to either shape distortion effects or nonextensivity effects. For example, anomalous diffusion described by nonlinear Fokker-Planck equations related to generalized entropies has been found to occur in nonextensive systems irrespective of the shape of steady-state distributions [24,40]. Based on the third choice thermostatistics developed in the present work, future studies may take advantage of the possibility provided by the Sharma-Mittal entropy to distinguish between phenomena related to nonextensivity and phenomena related to power-law distributions. Furthermore, the Sharma-Mittal entropy may worth being studied in its own right because it is the only entropy with a kernel described by a power law that belongs to the class of composable entropies of lowest order and gives rise to a third choice thermostatistics based on a partition function.

Appendix A: Energy obtained from the partition function

The crucial step in deriving equation (21) is to calculate the partial derivative $\partial Z / \partial \beta$. From equation (12) we obtain

$$Z=\sum_{i=1}^{N}\left[1-\frac{(1-s)\beta\,(\epsilon_i-U)}{Z^{1-q}}\right]_+^{1/(1-s)}\cdot \qquad (A.1)$$

Differentiating with respect to β yields

$$\frac{\partial Z}{\partial\beta}=-Z^s\sum_{i=1}^{N}p_i^s \times\left\{\frac{\epsilon_i-U}{Z^{1-q}}+\beta(\epsilon_i-U)\frac{\partial}{\partial\beta}\frac{1}{Z^{1-q}}-\frac{\beta}{Z^{1-q}}\frac{\partial U}{\partial\beta}\right\}\cdot \qquad (A.2)$$

Note that in equation (A.2) there is no need for the operator $[\ldots]_+$ because on account of the pre-factor p_i^s in equation (A.2) only those terms of the sum occurring in equation (A.1) are evaluated that are larger than zero. The first and second term in the sum of equation (A.2) vanish because of the constraint $\sum_{i=1}^N p_i^s\,\epsilon_i=U\sum_{i=1}^N p_i^s$. Then, by means of equation (14) we have

$$\frac{\partial Z}{\partial\beta}=\beta\frac{Z^s\sum_{i=1}^N p_i^s}{Z^{1-q}}\frac{\partial U}{\partial\beta}\quad\Rightarrow\quad Z^{-q}\frac{\partial Z}{\partial\beta}=\beta\frac{\partial U}{\partial\beta}\cdot \qquad (A.3)$$

Recall now that the generalized logarithm (15) satisfies for any differentiable function $f(x)>0$ the relation $\mathrm{d}\ln_q f(x)/\,\mathrm{d}x=f^{-q}\,\mathrm{d}f/\,\mathrm{d}x$. Consequently, from equation (A.3) it follows that

$$\frac{\partial}{\partial\beta}\ln_q Z=\beta\frac{\partial U}{\partial\beta}\cdot \qquad (A.4)$$

Substituting equation (18) into equation (A.4) gives us the result (21).

References

1. C. Beck, Physica A **295**, 195 (2001)
2. C. Beck, Physica A **305**, 209 (2002)
3. T. Arimitsu, N. Arimitsu, Physica A **295**, 177 (2001)
4. N.A. Salingaros, B.J. West, Environ. Plann. B **26**, 909 (1999)
5. R. Jain, S. Ramakumar, Physica A **273**, 476 (1999)
6. A. Upadhyaya, J.P. Rieu, J.A. Glazier, Y. Sawada, Physica A **293**, 549 (2001)
7. C. Tsallis, Braz. J. Phys. **29**, 1 (1999)
8. C. Renner, J. Peinke, R. Friedrich, Physica A **298**, 499 (2001)
9. E. Novikov, A. Novikov, D. Shannahoff-Khalsa, B. Schwartz, J. Wright, Phys. Rev. E **56**, R2387 (1997)
10. G.M. Viswanathan, V. Afanasyev, S.V. Buldyrev, S. Havlin, M.G.E. da Luz, E.P. Raposo, H.E. Stanley, Physica A **295**, 85 (2001)
11. M.G.E. da Luz, S.V. Buldyrev, S. Havlin, E.P. Rapsos, H.E. Stanley, G.M. Viswanathan, Physica A **295**, 89 (2001)
12. D. Prato, C. Tsallis, Phys. Rev. E **60**, 2398 (1999)
13. C. Tsallis, Phys. World **10**, 42 (1997)
14. M.F. Shlesinger, G.M. Zaslavsky, U. Frisch, *Lévy flights and related topics in physics* (Springer, Berlin, 1995)
15. P. Bak, *How Nature Works: the science of self-organized criticality* (Springer, New York, 1996)
16. L. Borland, Phys. Rev. E **57**, 6634 (1998)
17. L. Borland, F. Pennini, A.R. Plastino, A. Plastino, Eur. Phys. J. B **12**, 285 (1999)
18. A.G. Bashkirov, A.V. Vityazev, Physica A **277**, 136 (2000)
19. E.K. Lenzi, R.S. Mendes, L.R. da Silva, Physica A **280**, 337 (2000)
20. C. Giordano, A.R. Plastino, M. Casas, A. Plastino, Eur. Phys. J. B **22**, 361 (2001)
21. A. Renyi, *Probability theory* (North-Holland Publ. Company, Amsterdam, 1970)
22. C. Beck, F. Schlögl, *Thermodynamics of chaotic systems* (Cambridge University Press, Cambridge, 1993)
23. C. Tsallis, J. Stat. Phys. **52**, 479 (1988)
24. T.D. Frank, A. Daffertshofer, Physica A **285**, 351 (2000)
25. E.K. Lenzi, M.K. Lenzi, H. Belich, L.S. Lucena, Phys. Lett. A **292**, 315 (2002)
26. P.T. Landsberg, V. Vedral, Phys. Lett. A **247**, 211 (1998)
27. S. Abe, Y. Okamoto, *Nonextensive statistical mechanics and its applications* (Springer, Berlin, 2001)
28. C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A **261**, 534 (1998)
29. R.P. Di Sisto, S. Martinez, R.B. Orellana, A.R. Plastino, A. Plastino, Physica A **265**, 590 (1999)
30. B.D. Sharma, D.P. Mittal, J. Math. Sci. **10**, 28 (1975)
31. A. Katz, *Principles of statistical mechanics* (W.H. Freeman and Company, San Francisco, 1967)
32. H. Haken, *Information and self-organization* (Springer, Berlin, 1988)
33. A.R. Plastino, A. Plastino, Phys. Lett. A **174**, 384 (1993).
34. C. Tsallis, Physica A **221**, 227 (1995)
35. A. Plastino, A.R. Plastino, Phys. Lett. A **226**, 257 (1997)
36. R.S. Mendes, Physica A **242**, 299 (1997)
37. T. Yamano, Eur. Phys. J. B **18**, 103 (2000)
38. T.D. Frank, Phys. Lett. A **299**, 153 (2002)
39. T.D. Frank, A. Daffertshofer, Physica A **292**, 392 (2001)
40. T.D. Frank, J. Math. Phys. **43**, 344 (2002)
41. S. Abe, Phys. Lett. A **275**, 250 (2000)